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*Theoretical perspectives*

# Language in mathematics education – On the epistemic and reconstructivistic facet of languaging processes in linguistically heterogenous groups of learners

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**Abstrak** Artikel ini membahas proses berbahasa dalam pendidikan matematika dari perspektif teoritis dan metodologis. Bahasa bukan saja sebuah alat untuk pembelajaran bahasa tetapi juga sebuah medium yang sangat kompleks untuk menyampaikan makna. Bahasa memainkan peran penting untuk menjelaskan dan menumbuhkan serta merekonstruksi dan memaknai proses kognitif, tidak hanya dalam pendidikan matematika, tetapi karena karakateristik objek matematika yang abstrak sehingga peran bahasa diperlukan. Oleh sebab itu, bahasa sebagai sebuah dimensi mediasional sangat penting dalam proses pemahaman siswa dan peneliti baik itu dalam proses interpretasi atau dalam penjelasan dan tindakan yang diverbalkan, deiktis, baik eksplisit maupun implisit. Dalam artikel ini, perspektif dua sisi ini akan dijelaskan dengan memberikan wawasan ke dalam proses yang berhubungan dengan bahasa untuk menafsirkan dan memahami hubungan matematika dalam konteks Linkungan Belajar Substansial (*Substantial Learning Environments*, SLE) serta strategi relasional seperti Tugas Tambahan (*Auxiliary Task*).

**Keywords** *Bahasa dalam pendidikan matematika, Peran epistemik bahasa, Paradigma interpretatif*

**Abstract** In this article, languaging processes in mathematics education will be reflected from a theoretical and methodological viewpoint. Language is not just a tool for language learning: It is a highly complex medium for transporting *meaning*. It plays a key role in explaining and fostering as well as reconstructing and interpreting cognitive processes – not only in mathematics education, but due to the abstract nature of mathematical objects in a particularly important way. Thus, language as a mediational dimension is essential in the learners' as well as researchers' processes of understanding, be it in interpretational processes or in verbalized, deictical, either explicit or implicit explanations and actions. In this article, this dual-sided perspective will be explained by giving insights into language-related processes of interpreting and understanding mathematical relations in Substantial Learning Environments (SLE's) as well as relational strategies such as the 'Auxiliary Task'.

**Keywords** *Language in mathematics education, Epistemic role of language, Interpretative paradigm*

## Introduction: An overview on the role of language in mathematics education

Regarding the role of language for the learning and understanding of mathematical concepts, an important influence from a linguistic perspective comes from the early works of Vygotsky (1934): *Thinking-and-learning* processes are viewed as highly intertwined aspects of the human cognition in so far as that a) language *leads* to thinking processes and b) thinking is *articulated* and *encapsulated* in language (see Vygotsky, 1934, p. 218). The psychologist

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Vygotsky (1934) points out that speaking and thinking are two highly interrelated things since *"the relation of thought to word is not a thing but a process, a continual movement back and forth from thought to word and from word to thought."* (Vygotsky, 1934, p. 218), thus emphasizing the *cognitive/ epistemic function* of language and vice versa. As an example, he states how the Russian translator Krylow realized that the French word for 'grashopper' in La Fontaine's fable 'The Grasshopper and the Ant' was associated with a feminine gender, resulting in his decision to translate it as 'dragonfly' in Russian since it also has a feminine gender (whereas 'grashopper' is associated with a masculine gender in Russian) (see Vygotsky, 1934). Vygotsky's example of Krylows preliminary considerations shows, how the interrelation between thinking and speaking affects not only cognition in general but also culture-/ context-related processes of thinking and speaking between different languages and in nuances such as 'gender'. This dual-sided perspective on the interrelation between *language-and-cognition* as well as *language-and-culture/context* is an important theoretical basis for reflecting upon the (complex) mental processes between the three axes language, culture/ context and cognition (see figure 1).

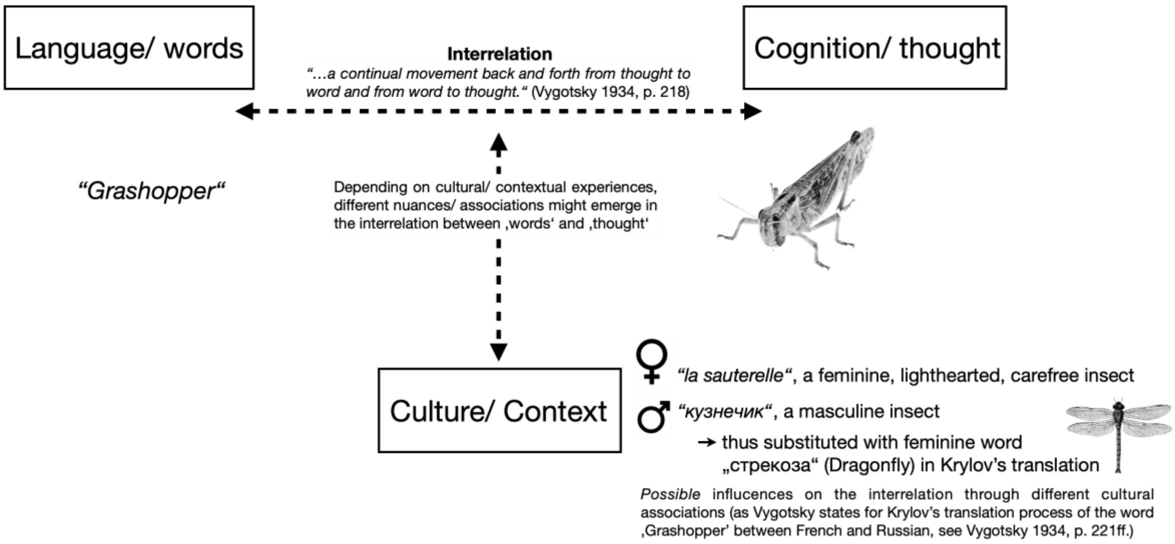


**Figure 1**. An illustration of the interrelation between words, thought and culture according to Vygotsky (1934) (Authors own elaboration)

Being a so-called 'social constructivist', Vygotsky (1934) emphasized not only *"the importance of culture and social context for cognitive development* […] *Vygotsky* […] *argues that learners can master concepts, which they cannot understand on their own, with help from instructors and peers."* (Gogus, 2012, p. 783; see also Vygotsky, 1978, 2004), thus in the 'Zone of Proximal Development'[2] (see Vygotsky, 1978). In contrast to the Piagetian hypothesis that children's development must precede their learning and that it is a highly or rather mainly individual process – in his late works, Piaget also emphasizes the importance of the social

[2] An often underemphasized, yet highly important level is the 'Actual Developmental Level': For a precise and viable fostering of learners in the 'Zone of Proximal Development', teachers have to assess what learners can do *without help* and only then tasks for fostering learners in the 'Zone of Proximal Development' can be designed, with *"problems that children cannot solve independently but only with assistance"* (Vygotsky, 1978, p. 33). Thus, Vygotsky (1978) does not only describe the necessity to foster learner's with cognitively demanding tasks, he also highlights the importance of a teacher's diagnostic and task-designing competencies.

dimension, e.g., by giving examples of how children co-constructively invent rules and norms (see Piaget, 1932) – , Vygotsky argues that *"learning is a necessary and universal aspect of the process of developing culturally organized, specifically human psychological function"* (Vygotsky, 1978, p. 90; see also Vygotsky, 2004). Furthermore, according to Vygotsky (1978), it is not possible to separate learning from its social and cultural context since learning is seen as the result of social interaction:

> *"Every function in the child's cultural development appears twice: first, on the social level and, later on, on the individual level; first, between people (interpsychological) and then inside the child (intrapsychological). This applies equally to voluntary attention, to logical memory, and to the formation of concepts. All the higher functions originate as actual relationships between individuals."* (Vygotsky, 1978, p. 57)

Following upon the Vygotskian perspective, Whorf formulated his hypothesis about the relation between language-use, thinking and (cultural) contexts – the so-called 'Sapir-Whorf-Hypothesis' – and described strong effects of (different) languages on the thinking of concepts such as "time" (see Whorf, 1940). This way of thinking in *hard* categories and differences is called *linguistic determinism* since it describes differences in is-or-is-not categories such as "there is no concept about x of time in language a" or "there is a concept about x in language b". This deterministic way of thinking about language-related differences was widely criticized as being not viable and became disproved (e.g., in Pinker, 1995), but the criticism did not lead to a total rejection of the Sapir-Whorf-Hypothesis, it rather led to the weaker form of the Sapir-Whorf-Hypothesis, the so-called *linguistic relativity*: Instead of assuming deterministic differences and hard categories, nuanced differences are seen as being possible between languages, affecting smaller categories such as 'associations with colors', 'gender-associations to words/ objects' or 'difference in order and nuances of thinking mathematical concepts such as fractions' (see Winawer et al., 2007; Thierry et al., 2009; Fausey & Boroditsky, 2011; Kuzu, 2023a).

Both perspectives, the Vygotskian as well as the Whorfian perspective, do emphasize the importance to reflect upon the *cognitive/ epistemic function* of language, thus upon the highly intertwined nature between 'words', 'thought' and 'culture/ context' (see figure 1). For this intertwined nature between speaking and thinking processes, the term 'languaging' was coined, meaning an *epistemic* or *cognitive function* of language for 'shaping' thinking processes beside the communicative function (Swain, 2006; Morek & Heller, 2012; Prediger & Şahin-Gür, 2020). An example for such a languaging-process would be that children hearing of 'triangles' for the first time, e.g., in kindergarten or other preschool institutions, start to realize and see triangles in their environment actively and start looking for objects with three corners and three sides, including an early process of category-building: What is a triangle, what is *not* a triangle, what are different shapes of triangles (e.g., some are more 'pointed' and others look more blunt) are discoverable (see Klose & Schreiber, 2018; Özçakır et al., 2019). But the relevance of language does not only stem from linguistic works: Early educational works in mathematics were also inspired from the psychological studies of Piaget, Aebli and others and important principles of mathematics education – like the *Operative Principle* (Wittmann, 1985), the *Principle of Cognitive Activation* (Kunter et al., 2013) and the *Register-Relation-Model* (Prediger et al., 2016) – are also highly language-related perspectives on mathematical learning processes. An important focus here lies on learning environments including a specific, cognitively stimulating language usage – *language* understood in a broader sense as all signs, manipulatives, words, gestures etc., including, but not being limited to mathematical signs (see Maier & Schwaiger, 1999; Steinbring, 2005; Riccomini et al., 2015; Meyer & Tiedemann, 2017; Barwell, 2020; Wessel, 2020; Robins & Crystal, 2021) – since learning always (also) means a language-mediated cognition process (see Kuzu, 2022).

This dual-sided theoretical perspective on language-mediated cognition processes in mathematics education will be explained in the following sections. This article does not claim to be exhaustive in showing all influences but aims at illustrating general theoretical aspects regarding the role of language in mathematics education.

## Language as a meaning-related cognitive tool: The role of language for fostering and explaining mathematical patterns, relations and concepts

Steiner (1973) speaks of mathematics as *thinking-education* and accordingly, Aebli (1973) states that,

> *"the thinking of the child does not only come from within and that the educator should not see his/ her role only in observing maturing processes [...], it rather is theoretically necessary and possible to foster learning processes in the thinking and behaviour of children and to influence his/ her development educationally.* […] *mathematical and pre-mathematical concepts and operations* […] *should gain a more vivid/ activated form and function in the thinking of children"* (cited in Steiner, 1973, p. XI; translated by the author).

Following such a demand for the psychological foundation of mathematics education with a stronger focus on thinking and learning processes, a lot of textbooks were remodeled in the German tradition of mathematics education after the 1970s insofar as that task structures aiming at so-called *perturbations*, meaning a *cognitive activation* or *cognitive conflicts* with regard to the conceptual understanding of mathematical objects or structures, became the main design principle of these tasks (see Meissner, 1986; Waxer & Morton, 2012). The language dimension was implicitly thought of in these cognitively activating tasks, but highly important since all mathematical impulses are accompanied by verbal explanations and hints: According to the *Operative Principle*, (see Wittmann, 1985) – a principle being based on the idea of *cognitively activating* learners through a step-wise process of examining mathematical objects – , mathematical relations have to be discovered and explained (see Wittmann, 2021, p. 149) and for being able to do so, the learners have to focus and examine a (mathematical) *object* at first, describe the *operations* being done on it (e.g., what was *added*, *taken away from the objects* etc.) and explain which *effect* was visible at the end (mostly explained through referring to relations, meanings etc.) (see Wittmann, 2021; see also Duval, 2006; Nührenbörger, 2015).

Here, language is of high relevance in at least two dimensions. First, on the *discoursive level* as a specific form of a *speech action*, meaning discourse practices or rather societal practices, thus *socially elaborated forms of communicative action* like 'explanations', 'descriptions' etc. (see Redder, 2008). 'Describing' the effects of operations on objects and 'explaining' them afterwards are two important and different forms of 'speech actions' that are necessary when analysing mathematical objects and relations according to the *operative principle*. 'Speech action' is a term being based on the Wittgensteinian philosophy of language, especially on his notion of 'language games' (see Wittgenstein, 1953): The notion 'language-game' is

> *"meant to bring into prominence the fact that the speaking of language is part of an activity, or of a form of life. Review the multiplicity of language-games in the following examples, and in others: Giving orders, and obeying them- Describing the appearance of an object, or giving its measurements- Constructing an object from a description (a drawing)- Reporting an event- Speculating about an event-"* (Wittgenstein, 1953, Part 1, Section 23).

Wittgenstein (1953) does not only give examples of everyday (speech) actions, inter alia he also gives examples of specific mathematical activities like

> *"Forming and testing a hypothesis- Presenting the results of an experiment in tables and diagrams- Making up a story; and reading it- Play-acting- Singing catches- Guessing riddles- Making a joke; telling it- Solving a problem in practical arithmetic- Translating from one language into another- Asking, thanking, cursing, greeting, praying."* (Wittgenstein, 1953, Part 1, Section 23).

Speech actions are 'societal practices' insofar as that they are not set universally, rather they are constructed socially and thus have to be according to what is expected by (other) individuals or groups of individuals as a viable 'mathematical explanation' etc. (see Redder, 2008; Erath et al., 2018). In educational contexts, norms concerning the quality and form of speech actions often vary from situation to situation, are often implicit and are mostly set by the teachers (see Esmonde, 2009; Stephan, 2014).

The second important level is the *conceptual meaning* level since the learners have to use language means referring to the *meaning of the object/ concept* as well as to *relations between mathematical objects* when analysing mathematical objects and relations according to the *operative principle*. Language *shapes* mathematical thinking processes, which is the so-called *epistemic function* of language (see Vygotsky, 1934; Wessel, 2020). Prediger et al. (2019) speak of so-called *meaning-related* language means as being typical for understanding mathematical concepts in the school/ academic language register, the latter being a register typical for school mathematical communication (see Prediger et al., 2019). A language register is a *„[…] set of meanings, the configuration of semantic patterns, that are typically drawn upon under the specific conditions, along with the words and structures"* (Halliday, 1978, S. 23). Some studies about language means in mathematics education do utilize a stronger linguistic approach for describing language means, being influenced, e.g., by a systemic functional linguistics approach and the register-notion of Halliday (1978) or by Cummins' (1979) differentiation between *basic interpersonal communicative skills* (BICS) and *cognitive academic language proficiency* (CALP) (see Pöhler et al., 2017; Wessel, 2020). In these studies, three typical forms of 'specific conditions' are often operationalized as 'registers': the *everyday-register* (being typical for informal situations and mostly consisting of contextualized and domain-specific language means), the *school-/ academic-register* (being typical for more formal situations – e.g., in institutions like schools with relevance across all school subjects – and mostly consisting of decontextualized and abstract language means) and the *technical-register* (being typical for condensed mathematical discourses and mostly consisting of highly abstract mathematical terms) (see Bauersfeld, 1980; Prediger et al., 2016; Wessel, 2020). Further approaches to language means in mathematics education utilize a stronger sociological approach, being influenced, e.g., by Bernstein's (1971) language analysis with regard to social class, (elaborated vs. restricted) codes and control dynamics between different social groups, by Bakhtin's (1975) 'heteroglossia' notion, which views language from a metalinguistic perspective as a 'social place' where different subject's worldviews enter into a power-related and conflict-laden dialogue or by Foucault's (1977) discourse-analytical approach with regard to implicit *"prescriptions that designate exclusions and choices*" (Foucault, 1977, p. 199, as cited in Norén, 2015) of subjects (see Straehler-Pohl & Gellert, 2013; Norén, 2015; Planas, 2018, 2021; Barwell, 2020; Kuzu, 2023c). Such a more sociological approach is, e.g., the operationalization of learners' linguistic resources as a facet of their 'social languages', where instead of a (static) register specific *"forms of language through which people and groups can be recognized and linked to specific worldviews in a given situation and context of culture"* are emphasized (Planas, 2018, p. 3-4).

Both perspectives – the more linguistic as well as the more sociological approach – have in common that language means have to be analyzed to reconstruct *latent meanings* of the language being used in learning processes as well as in linguistically heterogenous groups of learners, e.g., multilingual learners' linguistic resources and translanguaging processes or inclusion-exclusion processes through language (see Barwell, 2016, 2020; Planas, 2021; Meyer & Tiedemann, 2017; Uribe & Prediger, 2021; Kuzu, 2023a). A wholistic approach combining both approaches seems more appropriate to doing (socio-)critically aware research in mathematics education since especially a purely linguistic approach bears the risk of fading out important pedagogical, sociological as well as political implications of focusing language means from specific registers like the school-/ academic register: Often, privileged learners from academic families have more previous experiences in using language means from the school-/ academic register if compared to underprivileged learners, e.g., from communicating with their academic parents at the dining table, from 'training' speech actions like explanations, descriptions etc. from early childhood on (academic parents are often more aware of the importance of conducting speech actions at institutions in an expected way and thus do knowingly or unknowingly foster/ scaffold it), from having more (implicit) opportunities for learning complex language means by hearing their parents speak in full sentences or having them read books before sleep etc., and teachers, being academics, often tend to perceive and evaluate those learners as being more 'competent'[3], although a linguistic performance cannot be linked to cognitive potential automatically, which is why, following Bourdieu's influential sociological works (see, e.g., Bourdieu & Passeron, 1977; Bourdieu, 1991), a *socio-symbolic function* or rather power, especially with regard to the school-/ academic language register, is described critically in diverse studies (see Bourdieu, 1991; Snow & Uccelli, 2009; Morek & Heller, 2012). Bourdieu and Passeron (1977) describe the school-/ academic register as 'bourgeois parlance' and differentiate it from 'common

[3] 'Linguistic performance' is an important part of the bias-research on teachers: Teachers might tend to perceive students as more competent if they speak with less mistakes, if they speak in a non-mixed/ "clean" way with regard to language registers (e.g., by not mixing up language means from the everyday and school-/ academic register), if they conduct speech actions in expected ways etc., leading to an unnoticed privileging of monolingual's with an academic background due to social and habitual expectations to languaging-processes from an academic/ bourgeois viewpoint (since teachers are also academics), which at the same time may lead to a disadvantaging of emergent bilinguals/ second language learners due to the same reasons as well as the so-called 'monolingual mindset/ habitus' (see Gogolin, 1994; Jussim & Harber, 2005; Clyne, 2008; Barwell, 2009; Bonefeld & Dickhäuser, 2018; Umansky & Dumont, 2019). Bonefeld & Dickhäuser, (2018) could even reconstruct a biased grading of pre-service teachers in case of *the same* linguistic performance of monolingual and multilingual students: In their study, they let pre-service teachers evaluate identical texts of fictive students and only changed the name of the students (to a name from a non-migrational group as well as a name from a migrational group). They describe how the pre-service teachers evaluated the same text significantly better, if the name of the fictive student was from a non-migrational group (see Bonefeld & Dickhäuser, 2018).

parlance', being similar to Bernstein's restricted vs. elaborated code (1971) but with a stronger emphasis on the implicit, reproductional power to speaking such a 'bourgeois parlance'. In a Bakhtinean sense and with regard to what the students bring along as previous experiences from their academic/ bourgeois homes, this socio-symbolic power behind the school-/ academic language register can also be regarded as a speaking with 'alien words/ voices' since it is a speaking with the (powerful) 'words/ voices' of the academic parents, teachers etc. (see Bakhtin, 1975; Barwell, 2016). Ignoring such power-related dimensions of the school-/ academic-language registers – e.g., when designing a mathematics curriculum without explicitly mentioning this problematic facet and without giving instructions on how to balance it out in the process of assessing and evaluating students' performances in the context of their prior experiences[4] – would lead to an implicit inclusion of the social background to the assessment and evaluation of the learners' linguistic as well as cognitive performances. Bourdieu (1991) explains that the

> *"power of words is nothing other than the delegated power of the spokesperson, and his speech – that is, the substance of his discourse and, inseparably, his way of speaking – is no more than a testimony, and one among others, of the guarantee of delegation which is vested in him.* […] *By trying to understand the power of linguistic manifestations linguistically, by looking in language for the principle underlying the logic and effectiveness of the language of institution, one forgets that authority comes to language from outside* […] *Language at most represents this authority, manifests and symbolizes it.* […] *The stylistic features which characterize the language of priests, teachers and, more generally, all institutions, like routinization, stereotyping and neutralization, all stem from the position occupied in a competitive field by these persons entrusted with delegated authority."* (Bourdieu, 1991, p. 107-109).

However, this does not mean that researchers cannot identify and reflect upon the didactical implications of the relevance of language on learning mathematics, it rather means that – when trying to implement the language dimension into the assessment, evaluation and grading of students – teachers should rather use an *individual reference norm*, meaning an evaluation of learner's language learning processes according to their previous experiences and with regard to their specific background with a competence-oriented approach (e.g., for multilingualism) (see Barwell, 2009; Zahner & Moschkovich, 2011; Planas, 2021). Thus, if learners have an underprivileged background, a teacher should evaluate language learning processes, especially in the school-/ academic-register, with a higher awareness to the difficulty it bears for learners with none to little support from their families.

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[4] An example for such a problematic curriculum-implementation can be found in the mathematics-curriculum of the North-Rhine-Westfalia state in Germany: There, competences in the school-/ academic-language register are highlighted as an educational goal across all subjects ('Bildungssprache') without giving further instructions into how to assess, evaluate and differentiate between different learner groups (with or without pre-experiences in the academic-language register), thus treating all students as linguistic 'tabula rasas' or as having the same 'starting point'. German sociologists like A. Pant and A. El-Mafaalani criticize such a 'vagueness' as highly problematic since it then becomes a facet being relevant for students' assessment, evaluation and grading in all subjects (and in mathematics education, this might result in a 'focus problem': Linguistic performance might become more important than mathematical understanding, see Barwell, 2009; Planas & Setati, 2014), and this leads, as a consequence to the 'vagueness', to a dominance of the *social reference norm* (see El-Mafaalani, 2012; Pant, 2020; Atzert et al., 2020), meaning an undifferentiated comparison of all learners with each other, benefiting privileged learners from academic households and disadvantaging underprivileged learners, thus reproducing social and educational injustice.

### *Didactical implications of the mediation of mathematical understanding through language*

The relevance of language means on the meaning-related, conceptual level can be illustrated through the following example: The fraction 2/3 is a symbolic representation and it is not sufficient to write down the fraction or the highly condensed formal language ("numerator" and "denominator") to *understand* it. Learners have to think and explain the *meaning* adequately and for that purpose, ideally, they have to understand:

- *the relation between the 'part' and the 'whole'*: The 'part' has to be viewed as an object being integrated in a 'whole'. The verbalizing of this *relation* is not easy and learners tend to interpret both elements separately, which is a viable view on natural numbers (ℕ), but not for rational numbers (ℚ), where a new form of number emerges when relating a part to the whole, or the numerator to the denominator (see Behr et al., 1983; Prediger, 2006)
- *the relation between the structuring of the 'whole'/ 'part' and the (abstract) 'whole'/ 'part':* The part consists of "2" pieces, but is *one* abstract part(-stripe), and the whole is structured in form of "3" pieces in total, but it is *one* abstract whole. This structural relation only becomes visible when comparing different fractions in the sense of the operative principle, meaning a sequence of fractions with different denominators, e.g., by drawing equally sized fraction bars where the denominator is increased by "1", which leads to a smaller size of the pieces but does not change the size of the whole fraction bar. The same structural relation is of relevance for the relation between the 'part' and the structuring of the part in pieces equivalent to the number given in the numerator: The number of pieces may vary, but it is always one part(-stripe) that is focused (at least for the *part-of-whole* concept, for an overview of diverse fractional concepts see Tunç-Pekkan, 2015).

In this short description of what students have to be able to explain to show a (viable) conceptual understanding of a fraction as a *part-of-whole* from a prescriptive perspective, a lot of *meaning-related language means* (see Prediger et al., 2019; Kuzu, 2019, 2023a) were used like "share/ part/ whole", "the structuring/ pieces of the whole", "the part in the whole", "size of pieces becoming smaller", "the length of the part(-stripe)" etc. These language means mostly come from the school/academic and technical language register and might be linked to diverse individual notions, as empirical insights show (see Glade & Prediger, 2017; Kuzu & Prediger, 2017; Kollhoff, 2022), and the main role of the teacher is to design tasks to give the students opportunities to relate (their) informal everyday language means with meaning-related language means as well as viable interpretations and to link those with the formal register, whereas graphical representations have an important 'bridging function' in this process since they are needed to make visible the effects of *operative operations* on these objects (see Duval, 2006). According to Ruwisch (2015), a *viable* understanding or interpretation is given, if it *"transport*[s] *mathematical principles and meanings, if* [it is] *expandable and can serve as a basis for further notions without making it necessary for the children to relearn every time"* (Ruwisch, 2015, p. 5, translation by author) and accordingly, a *non-viable* understanding is not expandable and cannot function as a basis for further notions/ (conceptual) developmental processes without relearning, whereas a *partially-viable* understanding consists of an intuitive conceptual understanding with some/ first expandable elements in it (see Fischbein, 1975; Ruwisch, 2015; Kuzu, 2023a).

Thus, students' usage of (meaning-related) language or their way of using graphical representations for explanations gives important diagnostic insights into viable, partially-viable or non-viable individual notions: Students may misinterpret a fraction in diverse ways when showing and explaining their interpretation in a graphical representation, e.g., by thinking of differently sized wholes instead of a same sized whole, but in every interpretation, there also may be (first) viable aspects, making it necessary to analyze in detail (see figure 2).

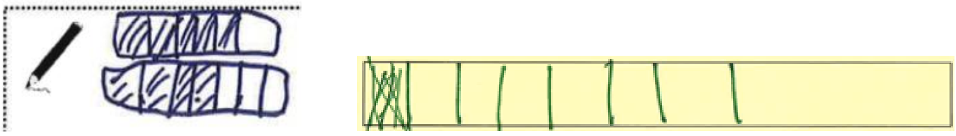
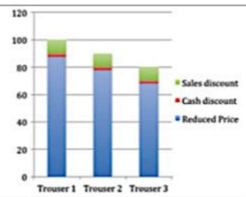
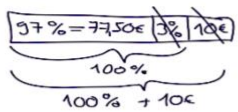


**Figure 2**. On the left side, Hakan compares the fractions 5/6 and 4/7, on the right side, Halim marks the fraction 1/8 (see Kuzu, 2019)

In figure 2, the student Hakan shows only a *partially-viable* interpretation of the "whole" on the left side: He may know how to draw a fraction into each bar and how to divide one bar into pieces as indicated in the denominator and that he needs one whole with equally sized pieces, but it is also visible that Hakan does not yet think of the wholes as *equally sized wholes* (see Kuzu, 2019, 2023a). On the right side, Halim marks 1/8 in a way that it does look like a non-viable understanding where he does not yet see necessity to split the whole into equally sized pieces (as indicated in the denominator), but when asked what he thought, he said that the fraction is *"the marked piece, but well, I drew it fast, I think the pieces are too small"*. Both cases shows that the *meaning* of a mathematical symbol, like a fraction, goes beyond a correct notation or calculation and that students have to understand complex and abstract aspects such as an equally sized whole, the relation of the part and the whole, splitting the whole into equally sized pieces etc. (see Kuzu, 2019, 2023a). Furthermore, students may think of a meaning or of first important aspects – like Halim – but that may only become visible when being explicitly asked for a verbalization of the thoughts.

From a didactical standpoint, it is thus important to pre-analyze conceptually meaning-related language means, to compare it to students' informal everyday language and to design learning environments based on these insights, which is the main idea behind a so-called *content-and-language-integrated approach* (see Pöhler & Prediger, 2015; Pöhler et al., 2017). In some cases, the same language means might be used with different meanings, like "piece" referring to the part in the everyday language register, whereas "piece" in the academic/ content-related context refers to the pieces in which the whole is split according to the denominator (see figure 2). In these cases, teachers have to foster a non-ambiguous language usage to prevent confusion. Here, in this process of differentiating and understanding a highly condensed mathematical object like a 'fraction' through decapsulating the meaning step-wisely and in its meaning-related components, a typical process of interpreting human-made (mathematical) signs is visible: It lies in the human nature to conserve and (re-)understand concepts and terms like "fractions" or "functions", which were encapsulated over generations and have to be explained (verbally) when decapsulating them (see Sfard, 1991; Steinbring, 2005, 2006; Prediger et al., 2019). Again, the *epistemic function* of language is crucial for understanding the meaning of those encapsulated concepts (see Carrutthers, 2002; Swain, 2006; Pöhler & Prediger, 2015): The conceptual understanding of a mathematical concept should always be accompanied by explanative discourses, where language from different registers may be used to understand and explain the meaning of a concept, as illustrated (figure 3) in Prediger et al. (2016).

<table>
<tr><th></th><th>Everyday register</th><th>School register</th><th>Technical register</th></tr>
<tr><td>Representation in words</td><td>Yesterday I was at a sale, in my favourite shop. The sale meant I received a 10 € discount for the trousers. Since I paid in cash, the sales clerk gave me another discount of 3 %. In all I only paid 77.50 €. How much was the original price?</td><td>In a sale, a pair of trousers was reduced by 10 €, and further 3 % discount was offered if you paid in cash. Hence, the final price was 77.50 €. What was the original price of the trousers?</td><td>If the original value is reduced by 10 € and then by 3 %, the new value is 77.50 €. What was the original value?</td></tr>
<tr><td>Graphical representation</td><td>
</td><td>
</td><td>
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<tr><td>Symbolic-numerical representation</td><td>–</td><td><table>
<tr><th></th><th>Original price (€)</th><th>Price with sales discount (€)</th><th>Price with 2nd discount (€)</th></tr>
<tr><td>Trouser 1</td><td>100</td><td>90</td><td>87.30</td></tr>
<tr><td>Trouser 2</td><td>90</td><td>81</td><td>78.57</td></tr>
<tr><td>Trouser 3</td><td>80</td><td>72</td><td>69.84</td></tr>
</table></td><td><table>
<tr><th></th><th>Base value</th><th>First reduction</th><th>Second reduction</th></tr>
<tr><td>Trial 1</td><td>100</td><td>90</td><td>87.3</td></tr>
<tr><td>Trial 2</td><td>90</td><td>82</td><td>78.57</td></tr>
<tr><td>Trial 3</td><td>80</td><td>72</td><td>69.84</td></tr>
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<tr><td>Symbolic- algebraic representation</td><td>–</td><td>–</td><td>$(G-10€)\times 0.97=77.50€$</td></tr>
</table>

**Figure 3**. Different language registers and examples (Prediger et al., 2016)

In figure 3, different language means in different registers are described, all of them serving the purpose of conceptually understanding percentages. Every register and representation may be useful for understanding and consolidating the mathematical concept, if allowed and activated in appropriate tasks, and it is developing throughout the registers (starting with everyday contexts and graphical representations and leading to more and more complex contexts and graphical representations). Learners' understanding of mathematical concepts can be fostered more efficiently, if the teachers think about concept related language means *beforehand* and use these for macro-scaffolding– meaning a usage of language means in different registers as in-designed elements in the tasks and task succession – , as well as in learner-teacher interactions, this as micro-scaffolding strategies (see Hammond & Gibbons, 2005; Pöhler & Prediger, 2015). Design-based research studies utilizing such a concept-and-language-integrated approach show how effective conceptual developmental processes can be fostered through such a planned and reflected language use (see Pöhler & Prediger, 2015; Kuzu, 2019; Wessel, 2020). Not only an *offering* of language means from different registers, but also the succession and the reflection of those language means have to be considered: It may help the learners to speak about the "whole stripe" (whole) and "download-stripes" (part) in a Download context before shifting to more formal and condensed language means like "share", "part", "whole", "numerator", "denominator" and "fraction", although this is no primitive shift: The meaning of these language means has to be linked to a) the conceptual meaning (through register relation with manipulatives, especially through the reflection of conversional processes , see Duval, 2006) and b) between the verbal registers, meaning that learners have to relate their informal everyday language means with the mathematical and condensed formal language means (see Prediger et al., 2016).

The epistemic function of language and the usage of meaning-related language means is not only relevant for early childhood mathematical topics but for every mathematical content, e.g., when understanding *variables-as-unknown* or *-indeterminate* parameters (see Bardini et al., 2005): In a secondary school pattern-describing formula like “2n + 1”, the variable has to be understood as an *indeterminate element* since it is not a placeholder for a concrete or fixed number, but for a *change* of a pattern depending on the position etc. (see Radford, 2018). Students have to understand for what this specific variable “n” stands: It is not a “coincidence”, that such a variable exists or works in the formula, it has a meaning and there is a reason why it works. For explaining the meaning, meaning-related language plays an important role again: An indeterminate variable might express a stretching or compression or another specific change, being visible in the graphical register, and this has to be described and explained. Understanding the *meaning* is also important for flexibility and innovation in e.g., equation solving in secondary school level (see Star, 2005) or in the tertiary level, e.g., for engineering mathematics (see Altieri & Prediger, 2016).

### *Cognitive Activation and meaning-related language in ‘Substantial Learning Environments’*

For further systematizing the idea of learning environments with a *cognitive activation potential*, which are mediated through a *cognitively activating* and *meaning-related language use* as described in the prior section, the idea of ‘Substantial Learning Environments’ (SLE) was formulated for middle childhood contexts – and ‘Substantial Playing Environments’ (SPE) for early childhood contexts, with a stronger focus on authentic games (see Tubach & Nührenbörger, 2016) – , meaning the designing of specific task formats with *a low threshold* and a *multitude of possible mathematical discoveries* with regard to relations, patterns and structures between mathematical objects (Wittmann, 2021, p 191ff.). They are similar to problem-solving tasks, but more structured and highly adaptable to the operative principle: For the first approach to the task, often a procedural access is given, where learners have to calculate structured tasks (which is why SLEs are categorized as ‘productive practice’ tasks, see Wittmann, 2021, p. 235ff.), e.g., tasks where the numbers/ objects are increased systematically by “1”, and following upon this first low-threshold procedural access to the task, learners shall discover and *explain* regularities and patterns being visible in the calculations, e.g., the effect of the increasing of the object or a relation between specific focused objects. In this sense, SLE’s are scaffolding pre-algebraic thinking and mathematical pattern discovery processes whilst analyzing arithmetic relations, especially if so-called *operative proofs* are conducted, meaning argumentational explanations being based on generic examples with objects (either enactive manipulatives or iconic objects like dots, stripes etc.) and leading to generalizations of rules or relations (Nührenbörger & Schwarzkopf, 2015*;* Schwarzkopf et al., 2018; Wittmann, 2021; Kuzu, 2022). Cognitive impulses are a planned or at least anticipated part for the second phase of the tasks, for the explanation of discoveries, so that again *meaning-related language* has to be considered an important and integral part when describing and explaining effects, relations, conceptual elements etc.

SLE’s bear a high potential for *cognitive activation*, but this potential needs to be stimulated through:

- an intelligent task design, consisting of register relation processes between the verbal register, consisting of either informal or formal language means, and the graphical register to visualize complex relational aspects (see prior section),

- an initial stimulation of first calculation and discovery processes as well as further supporting measures such as cards with hints etc.,
- and an intelligent teacher behaviour, not pre-empting discoveries but supporting the students' own discoveries in a constructivistic sense (see Nührenbörger et al., 2016, p. 5ff.; Scherer, 2019; Wittmann, 2021, p. 191ff.).

SLEs, looking at first glance like procedural formats, always have a *deeper structure* going beyond mere calculation or drawing processes (if it is a SLE from a geometric context), and since the main didactical goal is to stimulate the learners to verbally *explain their mathematical discoveries*, the high relevance of language becomes visible: Explaining structures and patterns is not easy and learners need a precise language use consisting of mathematical language, signs and objects (like numbers, terms, counters etc.) as well as *meaning-related language means* for explaining those: Just condensed formal language is not enough to explain relations and structures, especially if learners have to explain their thinking to other learners. Nührenbörger (2015) stresses out that for the purpose of *verbalizing* discoveries, a register relation process in the context of student-student interactions is important since explaining structures and patterns can be very hard if only a purely arithmetic access is chosen and a switching into a graphical representation, e.g., a cardinal representation, might help in showing the relations (see Nührenbörger, 2015; Kuzu, 2022), but then again learners have to use language to refer to the representation, operations made on the (graphical) objects etc.: (Pre-)algebraic generalizations are highly language-dependent as the study of Schwarzkopf et al. (2018) shows: When explaining the equality of the sum in a SLE about calculation chains, the learners used specific language means referring to steps, relations and structural elements and could explain the structure of the equation without using alphanumerical symbols (see Schwarzkopf et al., 2018).

A typical example for a SLE is the 'number wall' (see Steinbring, 2006). It is a task format where numbers are structured in a vertical and horizontal order and in an additive as well as subtractive relation to each other: For moving vertically upwards in the number wall, students have to carry out an addition, for moving vertically downwards in the number wall, students have to carry out a subtraction (see figure 4).

The number wall in figure 4 is a number wall from the 3rd grade. After learning the rules of how to calculate in these number walls (ideally, students should have learned about number walls in prior classes and with less complex number walls, e.g., with number walls with two or three rows in 1st or 2nd grade), students can on the one side practice basic arithmetic calculations and on the other side start to reflect changes in the sense of the 'operative principle' (see figure 5).

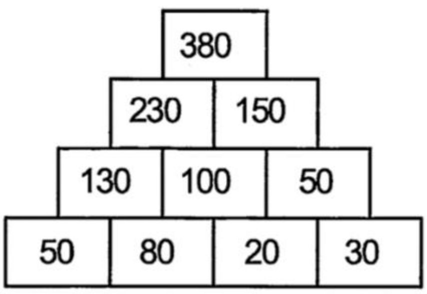



**Figure 4**. A four-storied 'number wall' (Steinbring, 2006)

In figure 5, students can reflect operative changes: e.g., they could examine how ‘objects’ (here: the numbers) are changed through ‘operations’ (here: Adding +10 to the second number from left in the first row of the number wall in the left) and which ‘effects’ are then visible (here: The fourth row/ highest number is exactly + 30 bigger than the fourth row in the number wall in figure 4). Steinbring (2006) highlights that the language being necessary for such a reflection process consists of much more than just ‘technical terms’:

> *“In classroom communications, it is very frequent to have recourse to the natural language-as in everyday life-as well as to possibilities of direct showing as essential means. Also in class, these means of communication mainly serve for exchanging and recording mathematical ideas. Descriptions in ordinary language, designations by means of striking names, direct showing and referring to something, all these are preliminary forms of interactively produced signs to code and develop aspects of mathematical concepts. In addition, technical terms, written notations, mathematical coding signs, variables, special mathematical symbols, diagrams etc. are the rather professional forms of mathematical signs.”* (Steinbring, 2006, p. 145)

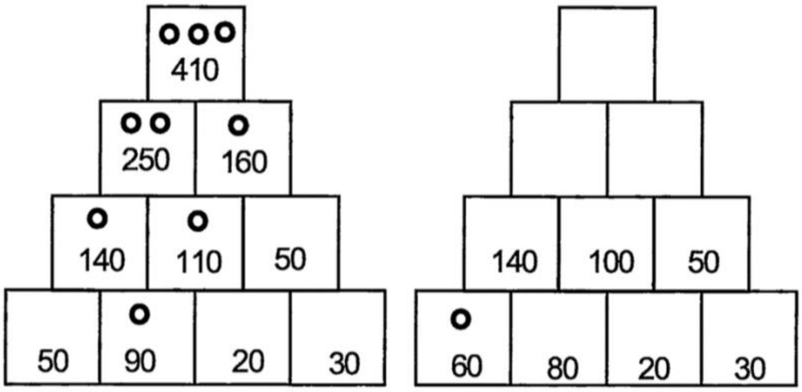



**Figure 5.** Operative changes in a four-storied ‘number wall’ (Steinbring, 2006)

Therefore, to explain *why* the number in the highest row is exactly + 30 bigger, students have to refer to the change of adding a tens to the second number in the first row (being represented here through a tens-dot in figure 5) and how such a change leads to the effect that the highest/ fourth row contains three tens. From the first to the second row, the tens is transmitted to second more numbers (140 and 110) and thus, in the third row, it is transmitted three times to two numbers (250 and 160), with “250” containing two tens-dots. For conducting such an *operative proof* to the question why the change in the fourth row/ highest number is three times bigger than the change in the first row, either enactive manipulatives of iconic objects are crucial since they mediate the abstract mathematical understanding and the languaging process (see Steinbring, 2006; Duval, 2006).

Such a sign-based and language-mediated reflection process, which goes beyond the mere calculation process, leads to a more flexible as well as analytical *thinking process*, it is what Steiner (1973) meant with ‘thinking education’ and what Aebli (1973) emphasized as a *more vivid/ activated form of thinking mathematical concepts and operations* (if compared to a purely calculation based approach, where no reflection process occurs). Teachers have to design SLE-tasks containing such ‘perturbative questions’, relations and structures, only then will a systematic mathematical reflection process be made possible. By continually reflecting and languaging the structures and relations in such SLE-tasks throughout the school years, students might develop a pre-algebraic understanding of numbers by understanding that they are interrelated and modifiable objects, where specific ‘operations’ have explainable ‘effects’, and that there are consistent, transferable ‘rules’ behind numerical relations. An example for such a discoverable rule is the ‘compensation rule’, which students can reflect in number walls with opposite operations (adding and taking away a tens from the numbers in the first row) and further reflect in, e.g., arithmetic strategies (see Steinweg et al., 2018; Kuzu, 2022). Another example for a SLE, intended for a use in primary school (1st or 2nd grade), is the ‘number chain’ (see figure 6).

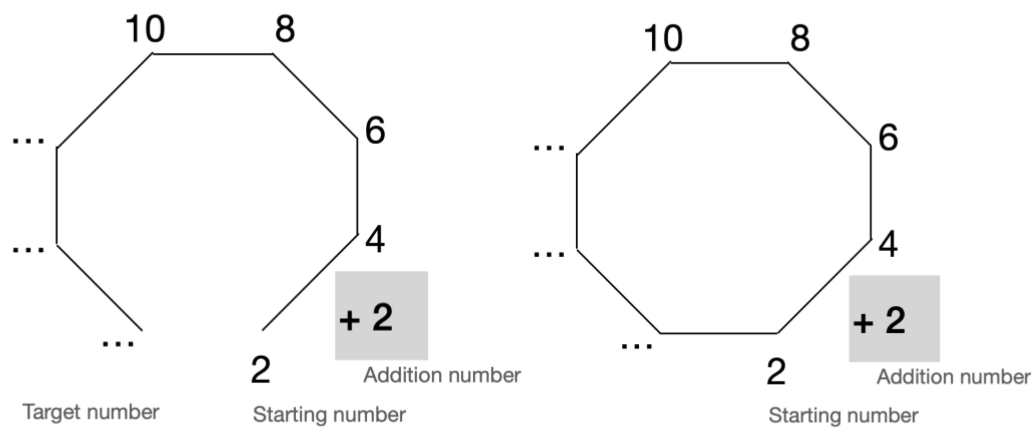



**Figure 6**. On the left side, a number chain in an Octagon-form, on the right side the intentionally wrong Octagon (authors own elaboration)

The number chain is arranged in an Octagon-form but there are multiple other possibilities for arranging number lines (e.g., as “straight” number lines or in other geometrical forms). An arrangement in an Octagon-form makes specific discoveries easier (e.g., the relation between the numbers “facing” each other), but before reflecting upon the structures and relations in this SLE, again, students have to learn how to calculate in this SLE in a first step: In an anti-clockwise manner, the missing corners are calculated through addition. After understanding the construction-norms of this SLE, students can be asked to explain what they could discover. If no discoveries could be made, a hint can be given through a hint-card, where diagonally opposing numbers are highlighted, while at the same time first subtraction tasks between these highlighted elements are listed and a cardinal register relation is offered (see figure 7).

After the hint, the learners may discover the pattern more easily: the diagonally opposing numbers always have a difference of “8”, if the smaller number is subtracted from the bigger number (but it also works, if the bigger number is subtracted from the smaller number, being interesting for a use in secondary school). This structure can be explained by looking at the relation between the diagonal numbers: In every step, the addition number is added and the diagonal numbers are four steps away from the opposed number, so that the addition number is added exactly four times. An explanation in an algebraic form might be possible in secondary school, but in primary school, a cardinal approach with an *operative proof* is more suitable: Visualizing the addition number with different colors, learners might see that with each step a specific addition number is added and that this number can be arranged in a rectangular form, where the multiplicative structure becomes visible and explainable.

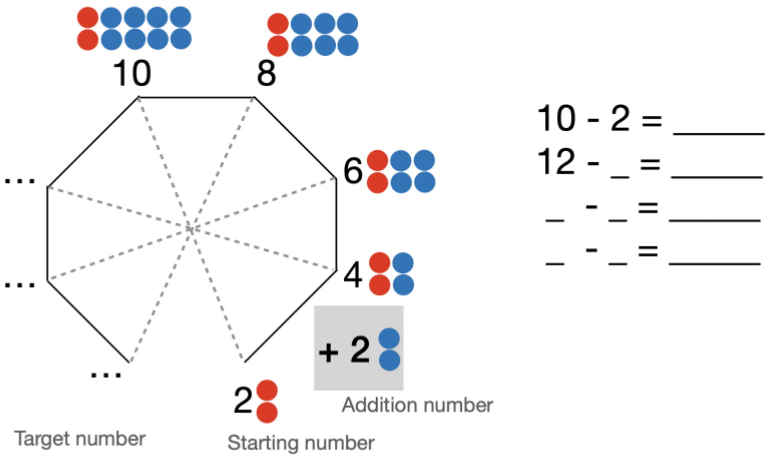



**Figure 7**. The hint-card being given to the students, where the diagonals are highlighted and cardinal dots are visible.

After explaining the structure, learners might test it for other relations: e.g., for octagons with another number of corners or with regard to further facets. In one case, a learner even saw a pattern in the point of intersection of the diagonals: He discovered that the value of that point must be equal since the difference between the diagonals was identical and thus he deducted that the diagonals must have the length of "8" with the intersection point being at "4". Further discoveries can be related to the sum of all corners etc. In figure 6, there is also a "special case" for perturbating the learners: It is an intentionally wrong octagon because the line between the starting number and the target number is wrong. It can be given to the students intentionally with the task to find the mistake, so the cognitive activation can be planned beforehand in form of a find-and-explain task. In all of these further activities, learners have to verbalize their discoveries through explaining the observed structures, ideally by relating registers. They can explain it by referring to "four times two objects/ counters" or "four steps with each having two dots added", the first explanation being from a more formal register and the latter being from a more everyday related register, but both explanations are correct.

Following this paradigm shift on the importance of a perturbative and language-sensitive task design, which was illustrated with two SLE-examples, research in mathematics education often focuses on the conceptual development of learners as well as the question *how* it can be fostered with designed, cognitively demanding tasks (in the 'Zone of Proximal Development', see Vygotsky, 1978), which is why mathematics education is called a 'design science' by Wittmann (1995). Methods like *design-based-research* tie upon this perspective, aiming at

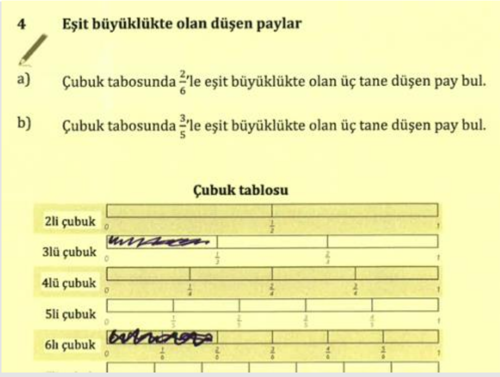
> *"(1) improving subject-matter classroom teaching by designing teaching-learning arrangements for a certain topic and (2) generating theoretical contributions by empirical research in order to understand the initiated teaching-learning processes for a certain topic […]. The resulting local theory serves as a rationale for the design and aims in the future for generalizations for further classroom contexts and topics."* (Prediger, 2019, p. 5).

## Reconstructing and interpreting through language: Insights into methodological issues and interpretational processes

(Epistemic) Language is not only important for explaining and fostering the mathematical understanding, it also is important for a reconstructional process from the researchers' perspective. Since in empirical research, no hypothesis can be stated without indication, the researcher needs empirically observable indications, like utterances, gestures etc. being conserved in form of (digital) data (Erickson, 2005; Mayring, 2015). As Piaget (1929) states, a researcher or observer needs a process of (verbal) communication to be able to reconstruct since *"many inexpressible thoughts must remain unknown so long we restrict ourselves to observing the child without talking to him"* (Piaget, 1929, p. 6-7; see also Ginsburg, 2009).

In educational practice, e.g., in schools, most interactional experiences lead to a fast and situational type of formative assessment but that is not fully reliable since it lies in human nature to misinterpret perceptions or incorrectly remember situations (Ginsburg, 2009, p. 111ff.; Lupyan, 2017). This also leads to a specific methodologial problem, even if the data is collected in form of videos: Since no reconstruction of understanding-processes through brain waves is possible, in empirical research utterances in transcriptions have to be analyzed carefully. The methodological issue here is that verbalized utterances *might*, but also *might not* represent actual thinking processes and that there are certain levels of (possibly) meant or not meant meanings depending on habitual and developmental aspects, e.g., one may assume that young children more often do say what they think – that they are more "honest" in this respect – and that older, middle or late childhood subjects become more used to divergent ways to say what they think (or not). The older the subjects are, the more they may know about strategic aspects, which have to be considered when concluding specific hypotheses about thinking processes based on utterances, like saying the opposite of what one thinks (e.g., to hide thoughts), how to verbalize it to meet the expectations or to create a specific impression, how to adapt the wording to the context etc. (see Ginsburg, 1997), but this doesn't mean that the interpretation of young(er) learners' utterances is easier, it has other obstacles: Utterances of younger children might be more ambiguous because of a not yet consolidated language use and the (higher) discrepancy between normative expectations in terms of a (appropriate) verbalization versus situation-based and descriptively reconstructable elements in students' utterances. Even if it is more reliable, if one analyzes data and does not only depend on perception, it still does not mean that the researcher grasps every (important) detail, as the following sequence from Kuzu (2019) illustrates:

**Context and task:** The multilingual students Halim and Hakan have to find equal shares in the fraction bar by looking at the length of the part-stripe (for further information on both students, the setting etc., see Kuzu 2019). The given share is 2/6 (see Task a on the right side), so that shares, students could possibly find, were 1/3, 4/12, 8/ 24 etc.



| | | |
|---|---|---|
| 12 | Teacher | Okay, now discuss your results, I will go to the other group and will come back in five minutes. Ah, and don't forget: Think about how you found the fractions! |
| 13 | Halim | Yeah… Ey - all right, what do you have written here *[looking at his fraction bar]* Oh, that is different. |
| 14 | Hakan | Yeah, mine is right. Look, they have the same length here. Yours are strange, what have you done? |
| 15 | Halim | Oh, well, I don't really understand it. What did you write down? *[looks at the task sheet of Hakan and then writes it on his own task sheet]* Thank you! |
| 16 | Teacher | *[coming back 3 minutes later]* Ok guys, what have you discussed? Who wants to explain what you did find out and how you did find it! *[Halim smiles and puts up his hands]* Halim, nice! |
| 17 | Halim | Yeah, well, *[reading his answer]* I looked at the end-, ending point of the part-stripes and then I look down vertically, that- that is how I find them! |
| 18 | Teacher | Nice, that is a very good answer! Thank you, Halim! |

An interpretative and extensive turn-by-turn analysis of this sequence, thus by interpreting each turn for itself and in the context of the prior turns, but not with regard to the following and normally covered turns – the method will be explained in more detail later in this section – begins with **turn 12**: The teacher seems to be in stress because of multiple groups he observes, he instructs the students to discuss their results and to *"think about how you found the fractions"*. Thus, the students seem to have solved the task at this point so that the instruction of the teacher is more a comparison task rather than a discussion task (a discussion would only occur if – after comparing the results – differences occur and the students feel the urge try to clarify the differences). In **turn 13** then, Halim states "Yeah", probably signalizing that he has understood the task or because it is anticipated as an expected behaviour (teachers often expect verbal feedback after giving instructions), and goes on with asking Hakan about his solution, possibly to compare his solutions or to copy Hakan's answer or parts of it. In **turn 14**, Hakan directly responds to Halim's comparison by stating that his solution is right, which indicates that both students have different solutions. The fact that he is a bit direct in his answer by stating that Halim's solution is "strange", asking him "what he has done" (probably surprised or not believing it), might mean that Hakan is very sure of his solution. It is possible that he is direct because of a positive or friendship-alike relation to Halim, but it could also be that he has a rather negative relation to him. In **turn 15** then, Halim's answers "Oh, well, I don't really understand it", which could be a defensive reflex to Hakan's direct answer to save his 'face' (see Zahner & Moschkovich, 2011) or it could be that he really did not understand the task. Halim goes on after that by writing down Hakan's answer, which rather confirms the hypothesis that he has not understood the task and also confirms the assumption from turn 13 that he looked at Hakan's task sheet/ answer to copy it. Another important detail is the fact that he says "Thank you" to Hakan after copying his answer. This indicates two things: First, Hakan does not react to Halim when he copies his answer, which might also be because he does not realize it, but since Halim says "Thank you" after being finished with copying the answer, it rather seems to be that Hakan knew of him copying his answer, there seems to be some kind of 'assistance/ support pact' between both learners (which tendentially confirms the first hypothesis from turn 14: That both learners have a rather positive relationship with each other, but it could also be a more utilitaristic, benefit-oriented behaviour). Following upon that, in **turn 16**, the teacher comes back to the group and immediately asks what the students discussed and who of them wants to explain what they found out, which might be a bit irritating since it is an ambiguous instruction: With regard to the necessary speech action, a discussion cannot be *explained*, it rather has to be *described*, and furthermore, a discussion is not a phase where something new is found out, the process of 'finding out something (new)' rather has to occur prior to a discussion. Halim seems to interpret the teacher's instruction as an instruction to read the solution to the task (instead of describing what happened in the discussion) since he starts to read his answer in **turn 17**. His answer in turn 17 is the answer he copied from Hakan in turn 15, thus he is intentionally deceiving the teacher, which indicates that being able to read a right answer is more important to him than being honest to the teacher, which might be because of a negative or not yet positive relationship with the teacher, a disturbance in the relationship to grownups or due to the fact that he is not yet aware of the consequences of intentionally deceiving others. It does not have to be an action out of "bad intention", it could also be a rash and strategical behaviour to get positive feedback, probably because he does not get positive feedback frequently. The teacher then gives highly positive feedback in **turn 18** by stating "nice", adding "very good answer" and thanking him personally.

What happens in this short sequence is typical for teacher-student and student-student interactions: Halim does not only copy the answer from Hakan, he acts strategically by doing what he is expected to do, he finds the solution and writes down an explanation and for this, he asks his mostly silent friend Hakan, who was a learner who showed one of the highest post- and follow-up-test achievements in the MuM-Multi study (see Kuzu, 2019, 2023a; Schüler-Meyer et al., 2019), while Halim himself was one of the learners who achieved the lowest score (see Kuzu, 2019). Until the end of the intervention, Halim presumably did not fully understand the conceptual meaning of a fraction, but later analyses showed that he had 'mastered' this strategy: He waited until the teacher was gone, asked his friend, noted down his answers or asked him, what he thought, then repeated it the moment the teacher came back (without understanding it in most cases). The teacher did not notice this strategy and thought until the end of the intervention that Halim was a learner who had understood the conceptual meaning of a fraction without any problems.

This short sequence illustrates that a certain degree of scepticism is important in empirical research in humanities *("Is that what the student says really what he thinks or what I interpret from his verbalization? Are there further possible interpretations")*, leading to the necessity of an 'interpretative awareness': An interviewer might have to ask more than once, what the subject meant, and has to take hidden, intended as well as non-intended facets into account (see Jungwirth, 2003; Schütte et al., 2019; Kuzu, 2023b). The latter means that *latent meanings* have to be analyzed, too: A text – understood in a broader sense as every form of expression, not only written, but also oral sequences, and as a sequence of language production – does always comprise of aspects *"apart from the intentions of those who produced it"* (Rosenthal, 2018, p. 18) – like implicit, unnamed but relevant political views and ideologies or specific presuppositions about rules, procedures, meanings etc. – , which is why a distinction between the (possibly) *"intended and the latent meaning"* (see Rosenthal, 2018, p. 18) is required. Generally, utterances and actions consist of more meaning than intended, because humans *"always produce more meaning than we are aware of at the moment of acting or speaking."* (Oevermann et al., 1979, p. 384). In mathematics education, these latent meanings may consist of conceptual elements the learners show without being aware of them (e.g., as concepts-*in-action* or theorems-*in-action* when children play with objects in the sense of a cardinal number concept, see

Vergnaud, 2009) or interaction-related aspects such as hierarchical structures with relevance to processes of knowledge-construction, like agency-patterns (see Norén, 2015), but even these latent elements need an indication, which is why the reconstruction of any kind of meaning, either (directly) visible or latent, is highly *language- or action-dependent* while at the same time the *interpretation* of the productions of research subjects underlies the same conditions: A researcher also has specific presuppositions about the texts that are known as well as not known to him:

> *"…researchers are subject to the same conditions that create this difference. Thus, implicit knowledge plays a role in the actions of the researchers, and can never be fully revealed or reflected on. They, too, habitually apply their knowledge without being aware that they are doing so, and reflecting on it requires making a considerable effort."* (Rosenthal, 2018, p. 19)

Thus, not only the learners but also the researchers are affected by the double nature of these methodological considerations: What is said, what is meant and what is visible without being (directly) said has to be part of the interpretative awareness or so-called *interpretative paradigm*, which is a term Wilson (1981) coined to distinguish empirical research in humanities from the sciences (where instead the *normative paradigm* is of relevance, e.g., in axiomatic subjects like Physics or pure Mathematics) (see Wilson, 1981). This does not mean that analyses are impossible because of scepticism, it means that analyses have to be conducted carefully by using an interpretative framework (diSessa, 2007). Such an interpretative framework, being based on the 'Objective Hermeneutics' (Oevermann et al., 1979) but with an abductive turn-by-turn-approach instead of a word-for-word-approach, is the interpretative 'Interaction analysis' (see Cobb & Bauersfeld, 1995; Krummheuer & Naujok, 1999; Jungwirth, 2003; Brandt & Schütte, 2010; Meyer, 2010; Brandt & Tiedemann, 2019; Kuzu, 2022, 2023a). Schütte et al. (2019) describe seven steps of this method:

1. Setting of the interactional unit (e.g., which classroom, grade, mathematical topic etc.),
2. Structure of the interactional unit (e.g., structuring after emerging topics, tasks etc.),
3. Displaying transcripts of selected sequences,
4. General description of each sequence,
5. **Detailed sequential interpretation of individual utterances,**
6. **Turn-by-turn analysis and**
7. Summary of the interpretation (see Schütte et al., 2019, p. 109ff.)

For conducting such an extensive turn-by-turn analysis – with steps 1- 4 being preparatory and steps 5 and 6 being the key steps of interpretation –, again *language* is crucial: The researchers' interpretations have to be related to observable/ reconstructable utterances and in this framework, a step-by-step procedure of making turns visible is followed: Turns or (short) sequences are made visible and interpreted successively (each turn is interpreted for itself and in the context of the prior turns, but not with regard to the following turns since the latter are covered and not yet visible when interpreting a specific turn) – mostly in small groups of interpreters – and might be interpreted with regard to earlier turns, but not with regard to turns not yet interpreted. By doing so, the different turn-related hypotheses do change/ develop in the course of the interpretational process and ideally, the analysis results in (more) consolidated hypotheses (see Cobb & Bauersfeld, 1995; Jungwirth, 2003; Schütte et al., 2019). These hypotheses are so-called 'abductions' – going back to the early works of Peirce on hypothetical reasoning (see Meyer, 2010; Paavola, 2011) – , meaning *"the generation of plausible explaining hypotheses regarding newly observed phenomena from within the data"* (Kuzu, 2023a, p. 12): Thus, every abduction has to be linked to utterances or utterance parts, has to be formulated carefully as an assumption or rather possible hypothesis. In the analysed sequence from turn 12 to 18, a particular emphasis was placed on a careful language usage: No strong attributions were made, only hypotheses in form of assumptions were intended and thus, only words indicating this interpretative 'mindset' or rather 'wordview' (Jungwirth, 2003) were used: "seems" (see interpretations of turn 12, 15, 16), "probably" (see interpretations of turn 13, 14), "or" standing for multiple hypotheses (see interpretations of turn 13, 15, 17), "indicating" (see interpretations of turn 15, 17), "might" (see interpretations of turn 14, 15, 16, 17), "possible" (see interpretations of turn 14), "could also be" (see interpretations of turn 14, 15, 17), and "assumption" (see interpretations of turn 15). No hypothesis can go beyond empirical indications (e.g., when an interpreter states that he *knows* what the student thinks because of his experience in school). In the course of the analysis, specific hypotheses might get expanded, rejected, or differentiated with each analyzed turn but, again, this can only happen based on empirical indications, thus through reference to visible/ observable parts in the interaction and its transcription (see Oevermann et al., 1979; Cobb & Bauersfeld, 1995; Krummheuer & Naujok, 1999). Peirce (1903) describes the process of abducting as follows:

> *"A mass of facts is before us. We go through them. We examine them. We find them a confused snarl, an impenetrable jungle. We are unable to hold them in our minds. We endeavour to set them down upon a paper; but they seem to be so multiplex intricate that we can neither satisfy ourselves that what we have set down represents the facts, nor can we get any clear idea of what it is that we have set down. But suddenly, while we are poring over our digest of the facts and are endeavouring to set them into order, it occurs to us that if we were to assume something to be true that we do not know to be true, these facts would arrange themselves luminously. That is abduction."* (Peirce, 1903, p. 282-283; as cited in Meyer, 2010)

An abductive reasoning must start from the data – or rather from the 'facts' – without having a specific theory or (prior) hypothesis in mind, starting with moments of 'surprises' a researcher has to formulate his hypothesis as *possible* explanations of the observable interaction (see Krummheuer, 2002; Meyer, 2010; Schütte et al., 2019). Being similar to the 'Grounded Theory' approach in its strict data-orientation, an interpretative analysis of data results in hypotheses with regard to specific cases and processes but for a further systematization and consolidation of these hypotheses, a so-called 'comparative analysis' between the different cases and within the cases themselves is important: Local insights into processes have to be compared to (similar) local insights in further cases to be able to state a resemblance and similarity between the abductions, which ensures that the gained insights are not singular but a pattern in different cases (Krummheuer, 2002; see also Krummheuer & Brandt, 2001; Tiedemann, 2012; Kollhoff, 2021). An interplay of local abductions in respect to specific research interests – e.g., with regard to mathematical as well as linguistic research questions – and a broadening to the whole learner group to consolidate these abductions might have a circular form (see figure 8).

In figure 8, the interpretative analysis consists of two main steps: A turn-by-turn analysis being conducted in every transcripted interaction sequence and the extension of the analysis to all n = 14 learners of the study for a) consolidating the hypotheses from the turn-by-turn analysis and b) making further assumptions about the frequencies of occurence (see Kuzu, 2022).

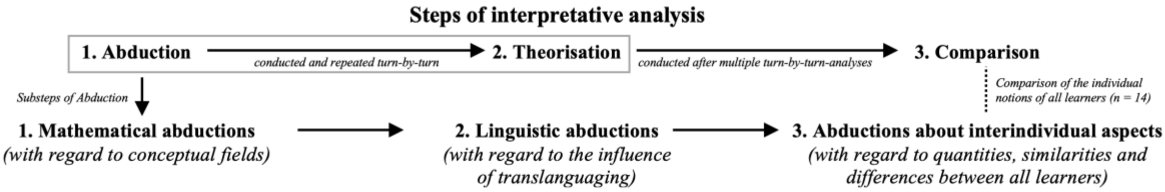
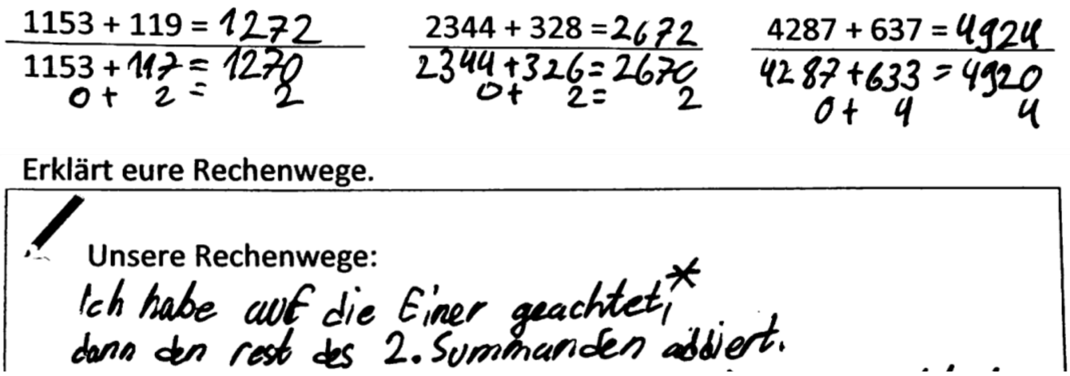



**Figure 8**. An example for an interpretative analysis-cycle consisting of the abductive as well as comparative analysis (Kuzu, 2023a)

Using such an interpretative framework is important for conducting careful analyses and generating *local theories*, meaning specific, topic-related consolidated hypotheses or insights into e.g., learning processes of specific mathematical objects, processes etc. (see Prediger et al., 2015). At the same time, this framework shows an adaptability to further in-depths analyses with mathematical tools such as the Toulmin-Scheme, Conceptual-Field-Analyses or with Epistemological Triangles as was applied in different studies (see Steinweg et al., 2018; Büscher, 2018; Kuzu, 2022, 2023a). A careful interpretative approach can also be used for analyzing written products in combination with transcriptions (see Brandt & Tiedemann, 2019). In such an analysis of a written product, it is necessary to show the genesis of the text because *only* the written products do not show how the text has emerged and the interpreter might miss important aspects: Jungwirth (2003) emphasizes the importance of partitioning a text into so-called 'Sinnabschnitte' (sequences of meaning), meaning the sequencing of an oral or written text into turns, chronological sequences etc. (see ibid.).

An example for an interpretative analysis of a written product and its genesis is presented in Kuzu, (2022). When generalizing their proceptual understanding of the so-called 'Auxiliary Task', which is a mental calculation strategy being based on the compensation rule (see Threlfall, 2002; Selter et al., 2012), students wrote down highly condensed answers. Written answers are often more condensed than oral answers and illustrate the Piagetian problem being mentioned in the prior section: If an interviewer does not talk to the child, the child's thoughts might not be visible. In the case of written answers, there are still thoughts visible (in a written form), but interpreting written explanations is highly complex since it is a condensed form of an answer, being *conceptually written* in most cases (see Koch & Oesterreicher, 1985; Ortmann & Dipper, 2019), meaning that an answer is more 'planned', elaborated and thus more complex in terms of sentence structure etc. while at the same time bearing the 'cost' of being a condensed product, where evolutionary aspects like the emergence of thoughts is (mostly) omitted or further ways of articulating, e.g. deicticals, are non-usable. Written answers are 'captured' like a statical photo, the motion leading to the momentum is not visible directly and has to be reconstructed through analyzing transcripts in parallel. Yet, there might be evolutional processes when writing answers, being triggered through typical interaction patterns in educational contexts: The interviewers or teachers might ask the students to be more precise and write down what they thought and how they mean it. Students then might write down their explanation in multiple phases or steps.

Such a case is described in Kuzu (2022), when a student wrote his answer successively in the course of the interaction with the teacher. In his first answer (step 1 of 2) the student wrote down his solution to a series of tasks and a first explanatory sentence (see Figure 9).



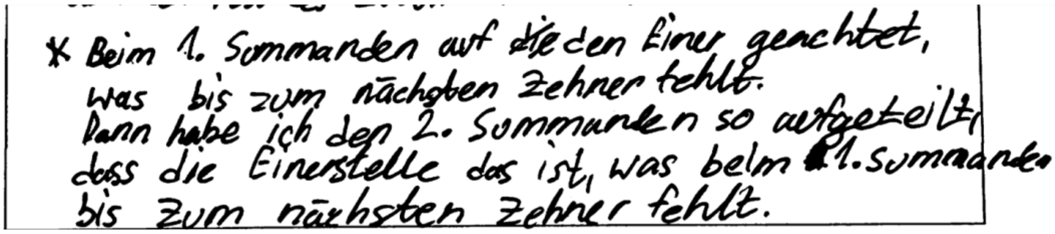

**Translation of the task:** *"Explain your way of calculation."* and *"Our calculation ways:"*
**Translation of the students' answer:** *"I looked at the ones, then the rest of the 2. summand added"*

**Figure 9**. A students written explanation of the 'Auxiliary Task' (step 1 of 2)

In figure 9, the student wrote down an answer by using a highly formal language: In terms of language means, it is a highly *encapsulated, formal* language, thus it is a very short answer. The translation of the German language mean "addiert" as the English counterpart "added" is not fully correct since in the German context, the word "added" represents the formal action being done for the calculation "addition", thus being connotated with a more abstract process, whereas another word, "hinzufügen", is less formal and means "adding to it" and should not be confused with it. Here, the student uses the highly formal "addiert", which is why it was translated as "added", although a word-creation like "additioned" might grasp the difference more precisely. This connotational problem shows the problem being described in the prior section: a *latent* meaning may stay hidden if a researcher skips the necessary commentary or does not reflect upon the occurring problem. This especially becomes visible in translational processes or in bilingual learning environments (see Kuzu, 2019), but is important for every interpretational process.

The teacher, at first not being able to understand what the student meant with his answer and solution, signalized him that she did not fully understand what he meant ("Can you write down what you meant, what you thought?"). That is also an important information the interpreter would miss if only the written product was analyzed. The students' solution is different from what was taught as a standard-type 'Auxiliary Task', it is a creative variation of it: The modification does not occur by rounding up the second number but by linking the ones of the second number with the ones in the first number and to correct the second number insofar as that the sum of the ones of both numbers is '10'. It is highly complex, but not wrong, and leads to a correct answer. Creatively divergent and non-standard-type solutions like these are not easy to interpret in a classroom-situation and teachers might not notice the creativity at once, which is probably the reason why the teacher asked for a further explanation.

After the teacher's question, the student then wrote down the star visible in his solution and added a second part to his written answer (step 2 of 2, see figure 10).




**Translation of the students' answer:** *"Looking at the ones of the 1. Summand, what was missing to the next tens. Then I split up the 2. Summand in a way that the position of the ones is that what is missing to the next tens at the 1. Summand."*

**Figure 10**. A student's written explanation of the 'Auxiliary Task' (step 2 of 2)

Here, the student gave a much more detailed answer. He described the process he thought of for solving the tasks by using *meaning-related* language means: He explained what he *thought of* when he looked at the ones ("…what was missing to the next tens") instead of capsulating it by solely saying "looked at the ones" as in figure 10. He expressed the *relation* between the numbers, especially between the ones: The ones of both numbers may be thought of as a sum of ten and that also works, if one compensates what was added. Thus, instead of rounding up for generating an easier task (which is why the strategy is called 'Auxiliary Task'), he developed another strategy for finding his own 'Auxiliary Tasks'. In his answer, he then went on by using further *meaning-related* language means by saying that he "split up" the ones in a way that he thought of "that what is missing to the next tens at the 1. Summand" and again made explicit a process that he encapsulated under the formal language mean "added" in his prior written answer (see figure 9). It is interesting to see how he explained what he meant when using the condensed words "looking at" and "added".

This brief analysis of a case study provides insights into a non-standard interpretation of the 'Auxiliary Task' and into the genesis of a creative solution process. These insights would be missing if the student had not been asked to explain his solution again and express his thoughts in greater detail and the teacher could have easily misunderstood the solution of the student, e.g., as a false strategy or wrong answer. Thus, teachers as well as researchers have to be cautious not to interpret too rashly since there might be implicit and not-articulated thoughts in the (formal) language use of students or connotational differences in translations which have to be considered if a researcher wants to analyze the *meaning* of student utterances and word uses. As Rosenthal (2018) states, researchers apply their knowledge without being aware of it, and reflecting upon such processes might be important for understanding what one might miss otherwise (see Rosenthal, 2018, p. 19). This does not mean that with enough reflection, a researcher might understand *everything*, but reflecting upon the meaning of what students say or write and searching for (more) explanations is a fundamentally important reflex in educational contexts since creative student solutions and (hidden) thoughts might stay hidden or – in the worst case – be easily misinterpreted.

## Summary: 'Language' as a crucial dimension for fostering students' conceptual understanding as well as a medium for understanding the understanding

In this paper, the double functionality of language from a theoretical perspective was described: The first function is related to *processes of understanding* in general and includes the question, how the thinking of learners is tied to language means and how language means being used in learning environments may foster the conceptual understanding. The tight and theoretically inseparable relation between thinking-and-speaking processes in a Vygotskian sense was grasped through using the term 'epistemic function of language', which is highly relevant for mathematical understanding processes (see Prediger & Şahin-Gür, 2020). In every utterance, in which learners explain their thinking of relations or concepts – when being *cognitively activated*–, this specific function becomes visible, e.g., when explaining mathematical relations in 'Substantial Learning Environments' (Wittmann, 2021). The 'epistemic function of language' is of relevance for every mathematical understanding, especially if learners use condensed language means and have to explain their thinking in teacher-student or student-student interactions (Nührenbörger, 2015). In such linguistic analyses of mathematical understanding, mostly language register models are used to conceptualize different types of language forms students may use, ranging from informal everyday language means to highly condensed formal language means (Prediger et al., 2016). Such an analysis of students' language usage and the (conceptual) meaning it bears is important for fostering learning process (see Pöhler & Prediger, 2015) as well as for understanding what was meant by students through reconstructing individual notions and thinking processes (see Büscher, 2018; Kuzu, 2019). The latter was illustrated in a case study of a written interpretation of the 'Auxiliary Task', where a student made explicit his implicit thoughts about a highly creative, non-standard strategy and explained his 'hidden' thoughts behind a short written explanation by adding further information on how he interpreted the Task by expanding his answer in a second step, where important meaning-related language means were added to make explain what he mentioned in a condensed form before.

At the same time, *language* is a necessary medium for empirical research in educational contexts such as mathematics education: Hypotheses about students' understanding or misunderstanding have to be indicated, but researchers as well as probands are subdued to highly subjective processes of interpretation, which is why a careful, step-wise and critical interpretation process is necessary (Schütte et al., 2019). Every interpretation has to be careful in terms of latent meanings and possible misinterpretations since human thinking does not follow a *normative* but *interpretative* paradigm, making it not impossible to analyze thinking processes but necessary to use a carefully conducted research using an interpretative framework (diSessa, 2007). This was also illustrated in the short analysis of the written explanation of the student: Latent meanings of mathematical terms in German (compared to similar English words with nuanced differences in meaning) as well as non-viable interpretations of the students' understanding, *if* the teacher did not ask for further explanations after step 1 (see figure 9 and 10), were discussed. What could not be deepened due to lack of space was the possibility to conduct a combined analysis through using the interpretative framework as well as in-depths analytical schemes, e.g., through the use of 'Epistemological Triangles' (Nührenbörger & Steinbring, 2009), but nevertheless, the necessity of generating *careful* hypotheses when interpreting language means with an epistemic function was described from a theoretical perspective as well as illustrated through a short empirical example.

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